

# String Corrections To The Riemann Curvature Tensor

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## Abstract

The string corrections to the Riemann Curvature tensor are found to first order in the string slope parameter, here proportional to  $\gamma$ . This is done for D=10 supergravity, the presumed low energy limit of string theory. We follow the perturbative approach. We also simplify a crucial result in our previous solution.

# 1 Introduction

In the past we have studied ten dimensional supergravity in superspace, [1,7]. We followed the perturbative approach of Gates et al. [2,3,5]. This group found a consistent solution to the Bianchi identities in superspace, thereby constructing a manifestly supersymmetric theory, achieving this to first order in  $\gamma$ , [5]. The theory yields a candidate for the low energy limit of string theory to that order. An alternative approach, given in [6] and [4], is the the Italian School approach. Furthermore, in [4], deformation techniques were used to study the torsion sector alone. The process to find a second order solution via the perturbative approach took many years. It was required to find the so called X tensor discussed in [2], and then finding a consistent set of solutions to the Bianchi identities in superspace in the torsion, curvature and H sectors at dimension zero through  $\frac{3}{2}$ . This was achieved in [1] and [7] to second order. Here we modify known results and find the string corrected Riemann curvature tensor. In particular we simplify a crucial result found in [7]. Our formalism is well discussed in [1] and [7]. However the following brief outline will help to make this paper self contained. The Bianchi identities in superspace are as usual,

$$[[\nabla_A, \nabla_B], \nabla_C] = 0 \quad (1)$$

Where

$$[\nabla_A, \nabla_B] = T_{AB}{}^C \nabla_C + \frac{1}{2} R_{AB}{}^{de} M_{ed} + i F_{AB}^I t_I \quad (2)$$

The  $t_I$  are the generators of the Yang-Mills gauge group. In order to illustrate the perturbative approach, we have the following results. The  $G$  tensor obeys

$$\nabla_{[A} G_{|BCD)} - T_{[AB}{}^E G_{E|CD)} = 0 \quad (3)$$

The Lorentz Chern Simons Superform obeys

$$\nabla_{[A} Q_{|BCD)} - T_{[AB}{}^E Q_{E|CD)} = R_{[AB}{}^{ef} R_{|CD)}{}^{ef} \quad (4)$$

The H tensor is defined as

$$H_{ADG} = G_{ADG} - \gamma Q_{ADG} - \beta Y_{ADG} \quad (5)$$

Here  $Y_{ADG}$  is the Yang Mills Superform, and  $\gamma$  is proportional to the string slope parameter.  $Q_{ADG}$  is the Lorentz Chern-Simons Superform. Here we set  $\beta = 0$ . Hence  $H_{BCD}$  obeys the identity

$$\nabla_{[A} H_{|BCD)} - T_{[AB}{}^E H_{E|CD)} = -\gamma R_{[AB}{}^{ef} R_{|CD)}{}^{ef} \quad (6)$$

This is the basis of the perturbative approach as we now can solve for torsions and curvatures as well as H sector tensors at successively higher orders in  $\gamma$ . That is, we first find solutions to (6). We then proceed to find mutually consistent solutions in the torsion

and curvature sectors. All of the required identities are listed in [1]. We write the order of the tensors in superscript as follows:

$$\begin{aligned} R_{ABde} &= R^{(0)}_{ABde} + R^{(1)}_{ABde} + R^{(2)}_{ABde} + \dots \\ T_{AD}{}^G &= T^{(0)}_{AD}{}^G + T^{(1)}_{AD}{}^G + T^{(2)}_{AD}{}^G + \dots \end{aligned} \quad (7)$$

In our previous work, [1], we found a candidate for the so called X tensor as discussed in [2]. This along with other techniques, enabled a solution to be found to second order. We proposed the following second order modification to the dimension zero torsion.

$$T^{(2)}_{\alpha\beta}{}^g = -\sigma^{pqref}_{\alpha\beta} X_{pqref}{}^g = -\frac{i\gamma}{6} \sigma^{pqref}_{\alpha\beta} H^{(0)g}_{ef} A^{(1)}_{pqr} \quad (8)$$

We maintained the definition of the supercurrent as used in [5],

$$A^{(1)}_{gef} = i\gamma \sigma_{gef\epsilon\tau} T^{mn\epsilon} T_{mn}{}^\tau \quad (9)$$

## 2 The Basic Analyses

The object  $R_{abmn}$  in ten dimensional superspace possesses all of the symmetries and properties of the Riemann curvature tensor. Hence, we will assume that it reduces to the Riemann tensor in ten dimensions and to the usual four dimensional tensor after dimensional reduction. We have from equation (1), the Bianchi identity

$$[[\nabla_{[\alpha}, \nabla_{\beta}\}, \nabla_a]\} = 0 \quad (10)$$

This generates the result

$$T_{\gamma[k]}{}^\lambda T_{\lambda[l]}{}^\tau + T_{\gamma[k}{}^g T_{g|l]}{}^\tau + T_{kl}{}^\lambda T_{\lambda\gamma}{}^\tau + T_{kl}{}^g T_{g\gamma}{}^\tau - \nabla_{[k} T_{l]\gamma}{}^\tau - R_{kl\gamma}{}^\tau - \nabla_\gamma T_{kl}{}^\tau = 0 \quad (11)$$

We will use this, along with some other results, to calculate the corrections to  $R_{abmn}$ . The required torsions to first order, which we select from the complete set listed in the appendix, are as follows:

$$\begin{aligned} T^{(0)}_{\alpha\beta}{}^g &= i\sigma_{\alpha\beta}{}^g; \quad T^{(1)}_{\alpha\beta}{}^g = 0; \quad T^{(0)}_{\alpha\beta}{}^\lambda = -[\delta_{(\alpha}{}^\lambda \delta_{|\beta)}{}^\delta + \sigma^g_{\alpha\beta} \sigma_g{}^{\lambda\delta}] \chi_\delta \\ T^{(1)}_{\alpha\beta}{}^\lambda &= 0; \quad T_{abc} = -2L_{abc}; \quad T^{(0)}_{\gamma gd} = T^{(1)}_{\gamma gd} = 0 \end{aligned} \quad (12)$$

These results were first found in [5]. We also maintain the following conventional constraint, at first order, as used in [5]. (This was later modified at second order,[1].)

$$T^{(1)}_{\alpha a}{}^\beta = -\frac{1}{48} \sigma_{a\alpha\lambda} \sigma^{pqr\beta\lambda} A_{pqr} \quad (13)$$

We recall that

$$R_{kl\gamma}{}^\tau = \frac{1}{4} R_{klmn} \sigma^{mn}{}_\gamma{}^\tau \quad (14)$$

And we also have the identity

$$\sigma^{mn}{}_{\gamma}{}^{\tau} \sigma_{rs\tau}{}^{\gamma} = -16\delta^m{}_{[r} \delta^n{}_{s]} \quad (15)$$

Hence we arrive at our basic equation for  $R_{klmn}$ .

$$R_{klmn} = -\frac{1}{8}\sigma_{mn\tau}{}^{\gamma} \{T_{\gamma[k]}{}^{\lambda} T_{\lambda|l]}{}^{\tau} + T_{\gamma[k]}{}^g T_{g|l]}{}^{\tau} + T_{kl}{}^{\lambda} T_{\lambda\gamma}{}^{\tau} + T_{kl}{}^g T_{g\gamma}{}^{\tau} - \nabla_{[k} T_{|l]\gamma}{}^{\tau} - \nabla_{\gamma} T_{kl}{}^{\tau}\} \quad (16)$$

This is valid to all orders. Substituting the torsions in (12) into (16) gives

$$R^{(0)}{}_{klmn} = -\frac{1}{8}\sigma_{mn\tau}{}^{\gamma} \{T_{kl}{}^{\lambda} T_{\lambda\gamma}{}^{\tau} - \nabla_{\gamma} T_{kl}{}^{\tau} |^{Order(0)}\} \quad (17)$$

$$R^{(1)}{}_{klmn} = -\frac{1}{8}\sigma_{mn\tau}{}^{\gamma} \{T_{kl}{}^g T_{g\gamma}{}^{\tau} - \nabla_{[k} T_{|l]\gamma}{}^{\tau} - \nabla_{\gamma} T_{kl}{}^{\tau} |^{Order(1)}\} \quad (18)$$

In reference [7], in order to close the curvature identity at dimension  $\frac{3}{2}$ , we required

$$\begin{aligned} R^{(1)}{}_{kl\gamma}{}^{\tau} = & +\frac{\gamma}{100} T_{kl}{}^{\tau} [T_{mn}{}^{\lambda} R^{(0)}{}_{\gamma\lambda}{}^{mn} + T^{(0)}{}_{mn}{}^g R^{(0)}{}_{\gamma g}{}^{mn} - \nabla_{\gamma} R^{(0)}{}_{mn}{}^{mn}] \\ & +\frac{1}{48} [T^{(0)}{}_{klg} \sigma^g{}_{\gamma\lambda} \sigma^{rst\lambda\tau} A^{(1)}{}_{rst} + 2\sigma_{[k|\gamma\lambda} \sigma^{rst\lambda\tau} \nabla_{|l]} A^{(1)}{}_{rst}] \end{aligned} \quad (19)$$

However we also have the Bianchi identity

$$T_{\alpha[a]}{}^X R_{X|b]de} + T_{ab}{}^X R_{X\alpha de} - \nabla_{[a} R_{|b]\alpha de} - \nabla_{\alpha} R_{abde} = 0 \quad (20)$$

Using (20) in (19) gives

$$R^{(1)}{}_{kl\gamma}{}^{\tau} = +\frac{i\gamma}{50} T_{kl}{}^{\tau} [\sigma^{[m]}{}_{\gamma\lambda} \nabla_m T_n^{[n]\lambda}] + \frac{1}{48} [T^{(0)}{}_{klg} \sigma^g{}_{\gamma\lambda} \sigma^{rst\lambda\tau} A^{(1)}{}_{rst} + 2\sigma_{[k|\gamma\lambda} \sigma^{rst\lambda\tau} \nabla_{|l]} A^{(1)}{}_{rst}] \quad (21)$$

The first term on the RHS of (19) and (21) is therefore shown to vanish. This now simplifies the result first found in [7], and hence greatly simplifies the calculation at third order. We find that (18) and (20) become

$$R^{(1)}{}_{kl\gamma}{}^{\tau} = +\frac{1}{48} [T^{(0)}{}_{klg} \sigma^g{}_{\gamma\lambda} \sigma^{rst\lambda\tau} A^{(1)}{}_{rst} + 2\sigma_{[k|\gamma\lambda} \sigma^{rst\lambda\tau} \nabla_{|l]} A^{(1)}{}_{rst}] \quad (22)$$

The use of the identity

$$\sigma^{pqr}{}^{\epsilon\tau} \sigma_{pqr}{}_{\mu\nu} = -48\delta^{\epsilon}{}_{[\mu} \delta^{\tau}{}_{\nu]} \quad (23)$$

as well as equation (9), simplifies (22) to

$$R^{(1)}{}_{klab} = \frac{i\gamma}{4} \sigma_{ab\tau}{}^{\gamma} \sigma^g{}_{\gamma\lambda} T_{klg} T_{mn}{}^{\lambda} T^{mn\tau} + \frac{i\gamma}{2} \sigma_{ab\tau}{}^{\gamma} \sigma_{[k|\gamma\lambda} T_{mn}{}^{\lambda} \nabla_{|l]} T^{mn\tau} \quad (24)$$

Hence, we have the first order corrections to the Riemann curvature tensor as required, or main result.

### 3 Conclusion

As is well known, the theory of strings, in a striking difference with respect to particles, involves a length scale, i.e. the string slope parameter, here proportional to  $\gamma$ . Supersymmetry and string theory have been realized to play a cooperative proactive role in several instances/scenarios, e.g. the landscape of supersymmetric compactifications, the AdS/CFT correspondence and the understanding of  $N = 4$  supersymmetric Yang-Mills (including exact results). Supergravity can be viewed as the low-energy limit of string theory. The latter possesses two expansion parameters, i.e. the string length, proportional to  $\gamma$ , which determines the corrections to supergravity results, and the string coupling constant, entering the string loop expansion. String theory naturally lives in ten dimensions, which naturally led to the revival of the Kaluza-Klein notion of compactification, with the well-known problematic implications in terms of moduli stabilization. The cosmological reflections of the theory led to the model by Kachru, Kallosh, Linde, Trivedi for the construction of supersymmetric compactifications [8]. In spite of the tremendous success of the model, a point deserves careful attention, connected to the loss of control of the ten dimensional solution and the need to involve classical and quantum effects in moduli stabilization. The latter may lead one to address the issue of the reliability of the treatment of non-perturbative corrections, connected to the spacetime being  $AdS_4$ . The fundamental reason for the difficulty in finding an explicit and simple de Sitter solution is inherent in the loss of supersymmetry, which requires solving the full supergravity equations of motion, rather than the much less involved task of imposing supersymmetry and solving Bianchi identities.

Based upon considerations such as the abovementioned ones, we revived the issue of providing a manifestly supersymmetric formulation of  $\gamma$  perturbed  $D=10$ ,  $N=1$  supergravity with fully consistent component-level symmetry transformation laws for the theory, to the relevant order in the  $\gamma$  perturbation. Based upon the results obtained in the present work, we can in a future work construct the corrected Ricci tensor and the corrected Scalar Curvature.

We simplified our previously published solution to second order by reducing (19) to (22). We then found the superspace expression for the string corrected Riemann curvature tensor, as it arises in ten dimensional perturbed supergravity, following the procedure outlined in equations (3) through (6). We now recall that the effective action can be written as an expansion in the parameter  $\gamma$ .

$$S_{eff} = \frac{1}{\kappa^2} \int d^{10}x e^{(-1)} [L_{(0)} + \sum_{n=1}^{n=\infty} (\gamma')^n L_{(n)}] \quad (25)$$

Let us suppose in the minimal scenario we consider the non supersymmetric low energy case of a string corrected General Relativity, which does not involve higher derivative gravity terms, nor the Gauss Bonnet term expected to arise from string corrected gravity, hence limiting our consideration simply to the Einstein Hilbert action. We might like to ask what minimal contributions result from constructing a curvature scalar using (24).

We now note that we can indeed proceed to higher orders by using the torsions listed in appendix, in conjunction with equation (16).

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## 4 Appendix: Main Results to Second Order

$$H_{\alpha\beta\gamma} = 0 + \text{order}(\gamma^2) \quad (1)$$

$$H^{(0)g}_{\alpha\beta} = \frac{i}{2}\sigma^g_{\alpha\beta}; \quad H^{(1)g}_{\alpha\beta} = 4i\gamma\sigma^p_{\alpha\beta}H^{(0)}_{pmn}H^{(0)gmn} \quad (2)$$

$$H^{(2)}_{\alpha\beta g} = +\sigma_{\alpha\beta}^g[8i\gamma H^{(0)}_{def}[L^{(1)g}_{ef} - \frac{1}{8}A^{(1)g}_{ef}] - \frac{i\gamma}{12}\sigma^{pqref}_{\alpha\beta}H^{(0)}_{def}A^{(1)}_{pqr}] \quad (3)$$

Where as reported in [5]

$$L_{gef} = H^{(0)}_{gef} + \gamma\{R^{(0)mn}_{[ef]}H^{(0)}_{[g]mn} + R^{(0)}_{[ef]}{}^{mn}H^{(0)}_{[g]mn} - \frac{8}{3}H^{(0)}_{pm[e]}H^{(0)p}_{|f|n}H^{(0)}_{|g]}{}^{mn}\} \quad (4)$$

$$T_{abc} = -2L_{abc} \quad (5)$$

Its spinor derivative is required to be, [5],

$$\nabla_{\alpha}L_{bcd}^{(0)} = \frac{i}{4}\sigma_{[b|\alpha\beta}T_{cd]}{}^{\beta}; \quad \nabla_{\alpha}L_{bcd}^{(1)} = i\gamma\sigma_{[b|\alpha\beta}T_{kl}{}^{\beta}R^{kl}_{|cd]} = R^{(1)}_{\alpha bde} \quad (6)$$

$$T^{(0)}_{\alpha\beta}{}^g = i\sigma_{\alpha\beta}^g = 2H^{(0)g}_{\alpha\beta}; \quad T^{(1)}_{\alpha\beta}{}^g = 0; \quad T^{(2)}_{\alpha\beta}{}^g = -\frac{i\gamma}{6}\sigma^{pqref}_{\alpha\beta}H^{(0)g}_{ef}A^{(1)}_{pqr} \quad (7)$$

$$T^{(0)}_{\alpha b}{}^{\gamma} = 0; \quad T^{(1)}_{\alpha b}{}^{\gamma} = -\frac{1}{48}\sigma_{b\alpha\phi}\sigma^{pqr\phi\gamma}A_{pqr}; \quad T^{(2)}_{\gamma g}{}^{\lambda} = 2\gamma\Omega^{(1)}_{\gamma gef}T^{ef\lambda} \quad (8)$$

Where

$$\Omega^{(1)}_{gef} = L^{(1)}_{gef} - \frac{1}{4}A^{(1)}_{gef}; \quad \Omega^{(1)}_{agef} = \nabla_{\gamma}\{L^{(1)}_{gef} - \frac{1}{4}A^{(1)}_{gef}\} \quad (9)$$

$$T^{(0)}_{\alpha\beta}{}^\gamma = -[\delta_{(\alpha}{}^\gamma \delta_{|\beta)}{}^\delta + \sigma^g_{\alpha\beta} \sigma_g{}^{\gamma\delta}] \chi_\delta; \quad T^{(1)}_{\alpha\beta}{}^\gamma = 0 \quad (10)$$

$$T^{(2)}_{\alpha\beta}{}^\lambda = -\frac{i\gamma}{12} \sigma^{pqref}_{\alpha\beta} A^{(1)}_{pqr} T_{ef}{}^\lambda \quad (11)$$

$$T^{(0)}_{\gamma gd} = T^{(1)}_{\gamma gd} = 0 \quad (12)$$

$$\begin{aligned} \sigma^g_{(\alpha\beta|} T^{(2)}_{|\gamma)gd} &= 4\gamma \sigma^g_{(\alpha\beta|} \Omega_{|\gamma)gef} H^{(0)}_d{}^{ef} - \frac{i\gamma}{6} \sigma^g_{(\alpha\beta|} \sigma^{pqre}_{g|\gamma)\phi} A^{(1)}_{pqr} T^{(0)}_{de}{}^\phi \\ &= \sigma^g_{(\alpha\beta|} X_{|\gamma)gd} \end{aligned} \quad (13)$$

Defining

$$\hat{O}^{ab\beta}_{gd\gamma} = [\frac{1}{2} \delta_{[g}{}^a \delta_{d]}{}^b \delta_\gamma{}^\beta - \frac{1}{12} \eta^{ab} \sigma_{gd\gamma}{}^\beta + \frac{1}{24} \delta_{[g}{}^{(a} \sigma_{d]\gamma}{}^{b)\beta}] \quad (14)$$

Therefore

$$T^{(2)}_{\gamma gd} = \hat{O}^{ab\beta}_{gd\gamma} [X_{\beta ab}] \quad (15)$$

$$R^{(0)}_{\alpha\beta de} = -2i\sigma^g_{\alpha\beta} H^{(0)}_{gde}; \quad R^{(1)}_{\alpha\beta de} = -2i\sigma^g_{\alpha\beta} [L^{(1)}_{gde} - \frac{1}{8} A^{(1)}_g{}^{ef}] + \frac{i}{24} \sigma^{pqrede}_{\alpha\beta} A_{pqr} \quad (16)$$

$$R^{(2)}_{\alpha\beta de} = -\frac{i\gamma}{12} \sigma^{pqref}_{\alpha\beta} A^{(1)}_{pqr} R_{efde}; \quad R^{(0)}_{\alpha bde} = -i\sigma_{[d|\alpha\phi} T_{b|e]}{}^f \quad (17)$$

$$R^{(1)}_{\alpha bde} = i\gamma \sigma_{[d|\alpha\phi} T_{kl}{}^\phi R^{kl}_{|de]}; \quad R^{(2)}_{\gamma gde} = 2\gamma \Omega^{(1)}_{\gamma gmn} R^{(0)mn}_{de} \quad (18)$$

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